

THE UNRAMIFIED BRAUER GROUP OF NORM ONE TORI

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ABSTRACT. Let k be a number field and K/k Galois. We transform the construction of the unramified Brauer group of the norm one torus $R_{K/k}^1(\mathbb{G}_m)$ into the construction of a special abelian extension over K . If $k = \mathbb{Q}$ and K/\mathbb{Q} biquadratic, we explicitly construct the unramified Brauer group of $R_{K/\mathbb{Q}}^1(\mathbb{G}_m)$.

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1. INTRODUCTION

Let T be a torus over a field k of characteristic zero, X a principal homogeneous space of T , and X^c a smooth compactification of X . Since the Brauer group $\mathrm{Br}(X^c) = H_{\acute{e}t}^2(X^c, \mathbb{G}_m)$ is a birational invariant of smooth proper varieties, which does not depend on the choice of X^c but only depends on X ; it is called the unramified Brauer group of X . Let $\mathrm{Br}_0(X^c)$ be the image of the natural map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(X^c)$. Formulas for $\mathrm{Br}(X^c)/\mathrm{Br}_0(X^c)$ can be found in [3]. Specially, if K/k is a Galois extension and $T = R_{K/k}^1(\mathbb{G}_m)$ its norm one torus, then $\mathrm{Br}(X^c)/\mathrm{Br}_0(X^c) \cong H^3(\mathrm{Gal}(K/k), \mathbb{Z})$. If K/k is cyclic, then $\mathrm{Br}(X^c) = \mathrm{Br}_0(X^c)$.

It is well known that the Brauer-Manin obstruction to the Hasse principle and weak approximation for rational points is the only one for X^c ([9]). To compute the Brauer-Manin obstruction, one need to construct the Brauer group. Recently, Colliot-Thélène ([2]) gave an explicit construction for a multi-norm torus of dimension 5. However, for general tori, it is still open, even for the norm one torus $R_{K/\mathbb{Q}}^1(\mathbb{G}_m)$, where K/\mathbb{Q} biquadratic.

The main aim of this article is to construct the unramified Brauer group for norm one tori $R_{K/k}^1(\mathbb{G}_m)$ with K/k Galois. In §2, we show any element in the unramified Brauer group of $R_{K/k}^1(\mathbb{G}_m)$ has a form from cup-product.

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Furthermore, we transform the construction of the unramified Brauer group into the construction of a special abelian extension over K (see Theorem 1). Some examples are also given in this section. In §3, using results of double covering of \mathbb{Q}^{ab} in [1, 6, 13], we give the explicit construction of the unramified Brauer group for the torus $R_{K/\mathbb{Q}}^1(\mathbb{G}_m)$, where K/\mathbb{Q} biquadratic.

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2. THE BRAUER GROUP OF $R_{K/k}^1(\mathbb{G}_m)$ WHEN K/k GALOIS

Let k be a field with $\text{char}(k) = 0$. Let K/k be a field extension and $T = R_{K/k}^1(\mathbb{G}_m)$ its norm one torus. Let X be the affine k -variety defined by $N_{K/k}(\Xi) = n \in k^\times$, which is a principal homogeneous space of T . And $\bar{k}[X]^\times / \bar{k}^\times \cong \hat{T}$ as a $\text{Gal}(\bar{k}/k)$ -module, where \hat{T} is the character group of T .

Let X^c be a smooth compactification of X . Let $\text{Br}(X)$ (resp. $\text{Br}(X^c)$) be the Brauer group of X (resp. X^c). Let $\text{Br}_0(X)$ (resp. $\text{Br}_0(X^c)$) be the image of $\text{Br}(k)$ in $\text{Br}(X)$ (resp. $\text{Br}_0(X^c)$).

By the Hochschild-Serre Spectral sequence, we have

$$0 \rightarrow H^1(k, \bar{k}[X]^\times) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(\bar{X})^{\text{Gal}(\bar{k}/k)} \rightarrow H^2(k, \bar{k}[X]^\times) \\ \rightarrow \text{Ker}[\text{Br}(X) \rightarrow \text{Br}(\bar{X})] \rightarrow H^1(k, \text{Pic}(\bar{X})).$$

Since $\bar{X} \cong \mathbb{G}_m^d$ over \bar{k} , it implies $\text{Pic}(\bar{X}) = 0$, where $d = [K : k] - 1$. Therefore

$$H^2(k, \bar{k}[X]^\times) \cong \text{Ker}[\text{Br}(X) \rightarrow \text{Br}(\bar{X})]. \quad (1)$$

Since X^c is geometrically rational, we have $\text{Br}(\bar{X}^c) = 0$. This implies the following lemma

Lemma 1. $\text{Br}(X^c)$ is contained in the image of $H^2(k, \bar{k}[X]^\times)$ in $\text{Br}(X)$.

Let $\Gamma_k = \text{Gal}(\bar{k}/k)$ and $\Gamma_K = \text{Gal}(\bar{k}/K)$. Denote $\mathbb{Z}[K/k] =: \mathbb{Z}[\Gamma_k/\Gamma_K]$. Let $i \geq 0$, we have the cup product

$$(\cdot, \cdot) : \mathbb{Z}[K/k] \times H^i(K, \mathbb{Z}) \rightarrow H^i(K, \mathbb{Z}[K/k]).$$

Let $\text{Cor}_{K/k}$ be the corestriction map $H^i(K, \cdot) \rightarrow H^i(k, \cdot)$.

Lemma 2. Let $\Gamma_K \in \Gamma_k/\Gamma_K$. Then $\text{Cor}_{K/k}(\Gamma_K, \cdot) : H^i(K, \mathbb{Z}) \rightarrow H^i(k, \mathbb{Z}[K/k])$ is the inverse map of the Shapiro's isomorphism $sh : H^i(k, \mathbb{Z}[K/k]) \rightarrow H^i(K, \mathbb{Z})$.

Proof. The case $i = 0$ is obvious. Then we only need to consider the case $i > 0$. Denote $g = \text{Cor}_{K/k}(\Gamma_K, \cdot)$. Since f is an isomorphism, we only need to show $g \cdot sh = id$ or $sh \cdot g = id$.

In the following, we will show that $g \cdot sh = id$. Let $C(\Gamma_k, \mathbb{Z}) \rightarrow \mathbb{Z}$ (resp. $C(\Gamma_k, \mathbb{Z}[K/k]) \rightarrow \mathbb{Z}[K/k]$) be a continuous Γ_k -module resolution of \mathbb{Z} (resp. $\mathbb{Z}[K/k]$). Suppose $x \in H^i(k, \mathbb{Z}[K/k])$, choose an i -cocycle $u \in C^i(\Gamma_k, \mathbb{Z}[K/k])$ which represents x . Hence

$$(g \cdot sh(u))(\sigma) = g(j_{\Gamma_K}(u(\sigma))) = \text{Cor}_{K/k}(j_{\Gamma_K}(u(\sigma))\Gamma_K) \\ = \sum_{\gamma\Gamma_K \in \Gamma_k/\Gamma_K} j_{\Gamma_K}(u(\gamma^{-1}\sigma))\gamma\Gamma_K,$$

where j_{Γ_K} is the projection $\mathbb{Z}[K/k] \rightarrow \mathbb{Z}$ by $\Gamma_K \mapsto 1, \gamma\Gamma_K \mapsto 0$ if $\gamma \notin \Gamma_K$. Similarly we can define $j_{\gamma\Gamma_K}$ for any $\gamma\Gamma_K \in \Gamma_k/\Gamma_K$. Since u is Γ_k -linear, we have

$$\begin{aligned} (g \cdot sh(u))(\sigma) &= \sum_{\gamma\Gamma_K \in \Gamma_k/\Gamma_K} j_{\Gamma_K}(\gamma^{-1}u(\sigma))\gamma\Gamma_K \\ &= \sum_{\gamma\Gamma_K \in \Gamma_k/\Gamma_K} j_{\gamma\Gamma_K}(u(\sigma))\gamma\Gamma_K = u(\sigma). \end{aligned}$$

□

In the reminder of this paper we always assume that k is a field of characteristic zero and $H^3(k, \mathbb{Z}) = 0$, eg. k is a number field or p -adic number field, see [7, Corollary 4.7]. The following lemma will show any element of $\text{Br}(X^c)$ has the form $\text{Cor}_{K/k}(\Xi, \chi)$.

Lemma 3. *Each element of $\text{Br}(X)/\text{Br}_0(X)$ in the image of $H^2(k, \bar{k}[X]^\times)$ is of the form $\text{Cor}_{K/k}(\Xi, \chi)$, where χ is a character of Γ_K and $\Xi \in K[X]^\times$ is a K -‘variable’. And $\text{Cor}_{K/k}(\Xi, \chi) = 0 \in \text{Br}(X)/\text{Br}_0(X)$ if and only if χ is the restriction of a character of $\text{Gal}(\bar{k}/k)$.*

Proof. Using the natural exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[K/k] \rightarrow \hat{T} \rightarrow 0,$$

and the assumption $H^3(k, \mathbb{Z}) = 0$, we deduce

$$H^2(k, \mathbb{Z}) \rightarrow H^2(k, \mathbb{Z}[K/k]) \xrightarrow{g} H^2(k, \hat{T}) \rightarrow H^3(k, \mathbb{Z}) = 0. \quad (2)$$

Define the $\text{Gal}(\bar{k}/k)$ -morphism $j : \mathbb{Z}[K/k] \rightarrow \bar{k}[X]^\times$ by $\Gamma_K \mapsto \Xi$. Then we have the maps $\mathbb{Z}[K/k] \rightarrow \bar{k}[X]^\times \rightarrow \bar{k}[X]^\times/\bar{k}^\times (\cong \hat{T})$. Hence the following diagram is commutative

$$\begin{array}{ccc} H^2(k, \mathbb{Z}[K/k]) & \rightarrow & H^2(k, \bar{k}[X]^\times) \\ & \searrow g & \downarrow f \\ & & H^2(k, \hat{T}). \end{array} \quad (3)$$

By the sequence (2), g is surjective. Hence f is also surjective. Write $\text{Ima}[\text{Br}(k)]$ to be the image of $\text{Br}(k)$ in $H^2(k, \bar{k}[X]^\times)$. There is the following isomorphism

$$H^2(k, \bar{k}[X]^\times)/\text{Ima}[\text{Br}(k)] \cong H^2(k, \hat{T}). \quad (4)$$

By Lemma 2 and the functoriality of cup product, we have the following commutative diagram

$$\begin{array}{ccccc} H^2(k, \mathbb{Z}) & \longrightarrow & H^2(K, \mathbb{Z}) & \longrightarrow & H^2(k, \bar{k}[X]^\times) \\ \downarrow = & \nearrow \text{Res}_{k/K} & \downarrow g_1 & \searrow g_2 & \downarrow = \\ H^2(k, \mathbb{Z}) & \longrightarrow & H^2(k, \mathbb{Z}[K/k]) & \longrightarrow & H^2(k, \bar{k}[X]^\times), \end{array}$$

where $g_1 = \text{Cor}_{K/k}(\Gamma_K, \cdot)$ is the inverse of the Shapiro’s isomorphism (by Lemma 2) and $g_2 = \text{Cor}_{K/k}(\Xi, \cdot)$. Then the above exact sequence (2) identifies with

$$H^2(k, \mathbb{Z}) \xrightarrow{\text{Res}_{k/K}} H^2(K, \mathbb{Z}) \xrightarrow{h} H^2(k, \bar{k}[X]^\times)/\text{Ima}[\text{Br}(k)] \rightarrow 0, \quad (5)$$

where $h = \rho \cdot \text{Cor}_{K/k}(\Xi, \cdot)$ and ρ is the quotient map

$$H^2(k, \bar{k}[X]^\times) \rightarrow H^2(k, \bar{k}[X]^\times)/\text{Ima}[\text{Br}(k)].$$

Using the canonical isomorphism $H^1(K, \mathbb{Q}/\mathbb{Z}) \cong H^2(K, \mathbb{Z})$, each element of $H^2(k, \bar{k}[X]^\times)/\text{Ima}[\text{Br}(k)]$ is of the form $\text{Cor}_{K/k}(\Xi, \chi)$ by (5), where χ is a character of Γ_K . Then each element of $\text{Br}(X)/\text{Br}_0(X)$ in the image of $H^2(k, \bar{k}[X]^\times)$ is of the form $\text{Cor}_{K/k}(\Xi, \chi)$.

Since the map $H^2(k, \bar{k}[X]^\times) \rightarrow \text{Br}(X)$ is injective by (1), the map

$$H^2(k, \bar{k}[X]^\times)/\text{Ima}[\text{Br}(k)] \rightarrow \text{Br}(X)/\text{Br}_0(X)$$

is also injective. By the exact sequence (5), we immediately have $\text{Cor}_{K/k}(\Xi, \chi)$ is zero in $\text{Br}(X)/\text{Br}_0(X)$ if and only if χ is the restriction of a character of $\text{Gal}(\bar{k}/k)$. \square

Suppose K/k is Galois. We say a field extension L/K satisfies the condition () if:*

L/F is abelian for any subfield $F \subset K$ satisfying K/F is cyclic.

Obviously L/K is abelian if L/K satisfies the condition ().*

Lemma 4. *Suppose K/k is Galois and L/K satisfies the condition (*). Then L/k is Galois and $\text{Gal}(L/K)$ is contained in the center of $\text{Gal}(L/k)$.*

Proof. We will show $\sigma(L) = L$ for any $\sigma \in \text{Gal}(\bar{k}/k)$. Let K^σ be the fixed subfield of σ in K . Obviously K/K^σ is cyclic, then we have L/K^σ is abelian by the condition (*). Therefore $\sigma(L) = L$. So L/k is Galois.

Let $\sigma \in \text{Gal}(L/K)$. For any $g \in \text{Gal}(L/k)$, let K^g be the subfield of K fixed by g . Hence K/K^g is cyclic. By the condition (*), $\text{Gal}(L/K^g)$ is abelian. This implies $\sigma g = g\sigma$ since $\sigma, g \in \text{Gal}(L/K^g)$. So σ is contained in the center of $\text{Gal}(L/k)$. \square

Theorem 1. *Suppose K/k is Galois. Let χ be a character of $\text{Gal}(\bar{k}/K)$. Then:*

- (1) *All elements of $\text{Br}(X^c)/\text{Br}_0(X^c)$ are of the form $\text{Cor}_{K/k}(\Xi, \chi)$.*
- (2) *$\text{Cor}_{K/k}(\Xi, \chi) \in \text{Br}(X^c)$ if and only if χ can factor through an abelian extension L/K which satisfies the condition (*).*

Proof. The part (1) follows from Lemma 1 and 3. So we only need to prove part (2)

Let $k(X)$ be the function field of X . Let A be a discrete valuation ring containing k with fraction field $k(X)$ and residue field κ_A . There is a residue map

$$\partial_A : \text{Br}(k(X)) \rightarrow H^1(\kappa_A, \mathbb{Q}/\mathbb{Z}).$$

By Grothendieck's purity theorem, we have

$$\text{Br}(X^c) = \bigcap_A \text{Ker}(\partial_A) \subset \text{Br}(k(X)),$$

where A runs through all above discrete valuation rings.

Suppose L/K satisfies the condition (*). Let χ be a character of $\text{Gal}(\bar{k}/K)$ which factors through $\text{Gal}(L/K)$. Let $\mathcal{B} = \text{Cor}_{K/k}(\Xi, \chi) \in \text{Br}(X)$, we will

prove $\mathcal{B} \in \text{Br}(X^c)$. Assume it is not, then there is a discrete valuation ring A such that

$$\partial_A(\mathcal{B}) \neq 0 \in H^1(\kappa_A, \mathbb{Q}/\mathbb{Z}).$$

Then there is an element $g \in \text{Gal}(\bar{\kappa}_A/\kappa_A)$ such that

$$\partial_A(\mathcal{B})(g) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

Since $k \subset \kappa_A$, we can fix an embedding $\bar{k} \hookrightarrow \bar{\kappa}_A$. Let K^g be the fixed field of g in K . Let f be the natural map $\text{Br}(X) \rightarrow \text{Br}(X_{K^g})$. Since $K^g(X) = K^g.k(X)$ is a finite unramified extension over $k(X)$, there is a discrete valuation ring $A_g \in K^g(X)$ which extends A . Then we have $\kappa_{A_g} = \kappa_A.K^g$ and $g \in \text{Gal}(\bar{\kappa}_A/\kappa_{A_g})$. By Proposition 1.1.1 in [5], we have

$$\partial_A(\mathcal{B})(g) = \partial_{A_g}(f(\mathcal{B}))(g).$$

Therefore $\partial_{A_g}(f(\mathcal{B})) \neq 0$.

Let

$$G = \text{Gal}(\bar{k}/k), U = \text{Gal}(\bar{k}/K^g) \text{ and } H = \text{Gal}(\bar{k}/K).$$

Choose a representation of double coset of G

$$G = \bigsqcup_{\sigma} U\sigma H. \quad (6)$$

By [8, Proposition 1.5.6], we have

$$\text{Res}_{G/U} \cdot \text{Cor}_{H/G} = \sum_{\sigma} \text{Cor}_{U \cap \sigma H \sigma^{-1}/U} \cdot \sigma_* \cdot \text{Res}_{H/H \cap \sigma U \sigma^{-1}},$$

where σ runs through all elements in (6). Note that H is a normal subgroup of G and contained in U , then we have

$$\text{Res}_{G/U} \cdot \text{Cor}_{H/G} = \sum_{\sigma} \text{Cor}_{H/U} \cdot \sigma_*.$$

Therefore

$$\begin{aligned} f(\mathcal{B}) &= \text{Res}_{G/U} \cdot \text{Cor}_{H/G}(\Xi, \chi) = \sum_{\sigma} \text{Cor}_{H/U} \cdot \sigma_*(\Xi, \chi) \\ &= \sum_{\sigma} \text{Cor}_{H/U}(\Xi^{\sigma}, \chi^{\sigma}). \end{aligned}$$

Since $\text{Gal}(L/K)$ is contained in the center of $\text{Gal}(L/k)$ by Lemma 4, we have

$$\chi^{\sigma} = \chi.$$

Hence

$$f(\mathcal{B}) = \sum_{\sigma} \text{Cor}_{H/U}(\Xi^{\sigma}, \chi).$$

Note that K/K^g is cyclic, we have $\text{Gal}(L/K^g)$ is abelian by the condition (*). Then we can choose a character $\hat{\chi}$ of $\text{Gal}(\bar{k}/K^g)$ which factors through $\text{Gal}(L/K^g)$ and lifts χ . Then

$$\begin{aligned} f(\mathcal{B}) &= \sum_{\sigma} \text{Cor}_{H/U}(\Xi^{\sigma}, \text{Res}_{U/H}(\hat{\chi})) = \sum_{\sigma} (N_{K/K^g}(\Xi^{\sigma}), \hat{\chi}) \\ &= (N_{K/k}(\Xi), \hat{\chi}) = (n, \hat{\chi}). \end{aligned}$$

Obviously

$$\partial_{A_g}(f(\mathcal{B})) = v_{A_g}(n)\hat{\chi} = 0,$$

which is a contradiction to $\partial_{A_g}(f(\mathcal{B})) \neq 0$. Therefore $\mathcal{B} \in \text{Br}(X^c)$.

On the other hand, suppose χ does not factor through any abelian extension over K which satisfies the condition (*). In the following we will show that $\mathcal{B} \notin \text{Br}(X^c)$.

Let L/K be the minimal abelian (cyclic) extension which χ factors through. Then there exists a subfield $F \subset K$ with $\text{Gal}(K/F)$ cyclic, such that L/F is not abelian. By the assumption there is a natural map $f : \text{Br}(X) \rightarrow \text{Br}(X_F)$. Let $U' = \text{Gal}(\bar{k}/F)$. Let G be the representation of double coset

$$G = \bigsqcup_{\sigma} U' \sigma H. \quad (7)$$

Similarly as above we have

$$f(\mathcal{B}) = \text{Res}_{G/U'} \cdot \text{Cor}_{H/G}(\Xi, \chi) = \sum_{\sigma} \text{Cor}_{H/U'}(\Xi^{\sigma}, \chi^{\sigma}).$$

It's clear that X_F is the affine variety defined by

$$\prod_{\sigma} N_{K/F}(\Xi^{\sigma}) = n,$$

where σ runs through all elements in the equation (7). Let X' be the affine variety over F defined by $N_{K/F}(\Xi') = n$. There is a close immersion $g : X' \rightarrow X_F$ defined by

$$\Xi' \leftarrow \Xi, \text{ otherwise } 1 \leftarrow \Xi^{\sigma}.$$

Hence

$$g^* f(\mathcal{B}) = \sum_{\sigma} g^* (\text{Cor}_{H/U'}(\Xi^{\sigma}, \chi^{\sigma})) = \text{Cor}_{H/U'}(\Xi', \chi),$$

where $g^* : \text{Br}(X_F) \rightarrow \text{Br}(X')$ is reduced by g .

By Lemma 3, we have $g^* f(\mathcal{B}) \neq 0 \in \text{Br}(X')/\text{Br}_0(X')$. Since K/F is cyclic, it implies $\text{Br}(X'^c) = \text{Br}_0(X'^c)$. Therefore $g^* f(\mathcal{B}) \notin \text{Br}(X'^c)$. So $\mathcal{B} \notin \text{Br}(X^c)$. \square

Let $K = k(\sqrt{-1}, \sqrt{d})$ and K/k of degree 4. Let $L = K(\sqrt[4]{d})$. It's clear that L/k is Galois (non-abelian) and of degree 8. Let χ be the unique nontrivial character of $\text{Gal}(\bar{k}/K)$ which factors through $\text{Gal}(L/K)$.

Proposition 1. *Let $K = k(\sqrt{-1}, \sqrt{d})$ and K/k of degree 4. The element $\text{Cor}_{K/k}(\Xi, \chi)$ is the unique generator of $\text{Br}(X^c)/\text{Br}_0(X^c)$.*

Proof. We know $\text{Br}(X^c)/\text{Br}_0(X^c) \cong H^3(\text{Gal}(K/k), \mathbb{Z}) \cong \mathbb{Z}/2$. This result follows from Theorem 1 and the fact that each group of order 4 is abelian. \square

We will use Proposition 1 to give an explicit example. Write $n = 2^{s_1} 17^{s_2} \cdot p_1^{e_1} \cdots p_g^{e_g} \in \mathbb{Q}^{\times}$. Let $n_1 = p_1^{e_1} \cdots p_g^{e_g}$. Let $D(n) = \{p_1, \dots, p_g\}$. Denote

$$\begin{aligned} D_1 &= \{p \in D(n) : \left(\frac{17}{p}\right) = \left(\frac{-17}{p}\right) = -1\} \\ D_2 &= \{p \in D(n) : \left(\frac{17}{p}\right) = \left(\frac{-1}{p}\right) = 1 \text{ and } \left(\frac{17}{p}\right)_4 = -1\}. \end{aligned}$$

Example 1. Let $K = \mathbb{Q}(\sqrt{-1}, \sqrt{17})$. Then the equation $N_{K/\mathbb{Q}}(\Xi) = n$ is solvable over \mathbb{Q} if and only if the following conditions hold:

- (1) $(-1, n_1)_2 = (\frac{n_1}{17}) = 1$; e_i is even when K/k is not totally split over p_i .
 (2) $(-1)^{\sum_{p_i \in D_1} e_i/2 + \sum_{p_i \in D_2} e_i} = (\frac{n_1}{17})_4$.

Let $K = k(\sqrt{d_1}, \sqrt{d_2})$ and K/k of degree 4. Suppose $x^2 - d_1 y^2 = d_2 z^2$ has a nonzero solution (x_0, y_0, z_0) in k . Let $L = K(\sqrt{x_0 + y_0 \sqrt{d_1}})$. We can see L/k is Galois (non-abelian) and of degree 8. Let χ be the unique nontrivial character of $\text{Gal}(\bar{k}/K)$ which factors through $\text{Gal}(L/K)$. Similarly as above we immediately have the following result:

Proposition 2. Let $K = k(\sqrt{d_1}, \sqrt{d_2})$ be as above. Then $\text{Cor}_{K/k}(\Xi, \chi)$ is the unique generator of $\text{Br}(X^c)/\text{Br}_0(X^c)$.

Finally we will use Proposition 2 to give an explicit example. Write $n = (-1)^{s_0} 2^{s_1} 13^{s_2} 17^{s_3} p_1^{e_1} \cdots p_g^{e_g} \in \mathbb{Q}^\times$. Let $D(n) = \{p_1, \dots, p_g\}$ and $n_1 = p_1^{e_1} \cdots p_g^{e_g}$. Let $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$. Obviously $15^2 - 13 \cdot 4^2 = 17$. Denote

$$D_1 = \{p \in D(n) : (\frac{13}{p}) = (\frac{17}{p}) = -1\}$$

$$D_2 = \{p \in D(n) : (\frac{13}{p}) = (\frac{17}{p}) = 1 \text{ and } (\frac{15 + 4\sqrt{13}}{p}) = -1\}.$$

Example 2. Let $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$. Then the equation $N_{K/\mathbb{Q}}(\Xi) = n$ is solvable over \mathbb{Q} if and only if the following conditions hold:

- (1) s_0 is even; $(\frac{n_1}{13}) \cdot (-1)^{s_1} = (\frac{n_1}{17}) = 1$; and e_i is even when K/k is not totally split over p_i .
 (2) $(-1)^{\sum_{p_i \in D_1} e_i/2 + \sum_{p_i \in D_2} e_i} = (-1)^{s_1 + s_2} \cdot (\frac{n_3}{2})$.

Remark 1. Let k be a number field and $K = k(\sqrt{a}, \sqrt{b})$ a biquadratic field. Sansuc ([10, Proposition 6]) gave a method to determine the existence of rational points for X or not. For each $n \in k^\times$, his method need to look for a rational solution $(\Xi_1, \Xi_2) \in k(\sqrt{a})^\times \times k(\sqrt{b})^\times$ of the equation

$$N_{k(\sqrt{a})/k}(\Xi_1) \cdot N_{k(\sqrt{b})/k}(\Xi_2) = n.$$

3. THE CASE THAT $k = \mathbb{Q}$ AND K/\mathbb{Q} BIQUADRATIC

In §3.1, we will recall some results of double covering of $\mathbb{Q}^{ab}/\mathbb{Q}$. In §3.2, the explicit construction for the biquadratic case will be given using Theorem 1 in §2 and double coverings in §3.1.

3.1. Double covering of \mathbb{Q}^{ab} . Suppose K/F is Galois. A double covering of K/F (defined in [6]) is an extension \hat{K}/K of degree ≤ 2 such that \hat{K}/F is Galois. Let \mathbb{Q}^{ab} be the maximal abelian extension of \mathbb{Q} . In the following we will describe all double coverings of \mathbb{Q}^{ab} (see [1, 6]) and of the cyclotomic field $\mathbb{Q}(\xi_n)$ (see [13]).

Let \mathcal{A} be the free abelian group on the symbols of the form $[a] (a \in \mathbb{Q})$ modulo the identifications

$$[a] = [b] \Leftrightarrow a - b \in \mathbb{Z}.$$

For all odd primes $p < q$, put

$$\mathbf{a}_{pq} = \sum_{i=1}^{\frac{p-1}{2}} \left(\left[\frac{i}{p} \right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{i}{pq} + \frac{k}{q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q} \right] - \sum_{l=0}^{\frac{p-1}{2}} \left[\frac{j}{pq} + \frac{l}{p} \right] \right)$$

and for prime $q > 2$, put

$$\mathbf{a}_{2q} = \left(\left[\frac{1}{4} \right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{k}{q} + \frac{1}{4q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q} \right] + \left[\frac{j}{q} - \frac{1}{2q} \right] - \left[\frac{j}{2q} \right] - \left[\frac{j}{2q} - \frac{1}{4q} \right] \right).$$

Let

$$\sin : \mathcal{A} \rightarrow \mathbb{Q}^{ab \times}$$

be the unique homomorphism such that

$$\sin[a] = \begin{cases} 2\sin(\pi a)(=|1 - e^{2\pi ia}|) & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0 \end{cases} \quad (a \in \mathbb{Q} \cap [0, 1)).$$

The composition field of all double coverings of \mathbb{Q}^{ab} is (see [1, main theorem])

$$\mathbb{Q}^{ab} \left(\{ \sqrt[4]{l} \}_{l:\text{prime}} \bigcup \{ \sqrt{\sin \mathbf{a}_{pq}} \}_{p,q:\text{prime}} \right).$$

Let $n \not\equiv 2 \pmod{4}$, we define a subset S_n of \mathbb{Z} associated to n as following:

- i) if $2 \nmid n$, set $S_n := \{\text{odd prime factors of } n\}$;
- ii) if $4 \mid n$ and $8 \nmid n$, set $S_n := \{-1\} \cup \{\text{odd prime factors of } n\}$;
- iii) if $8 \mid n$, set $S_n := \{-1, 2\} \cup \{\text{odd prime factors of } n\}$.

If $4 \mid n$, then for all $p, q \in S_n$ and $p < q$, we set

$$u_{pq} := \begin{cases} \sqrt{q} & \text{if } p = -1 \\ \sin \mathbf{a}_{pq} & \text{otherwise.} \end{cases}$$

If $2 \nmid n$, then for primes $p, q \in S_n$ and $p < q$, we set

$$u_{pq} := \begin{cases} \sin \mathbf{a}_{pq} & \text{if } p \equiv q \equiv 1 \pmod{4} \\ \sqrt{p} \cdot \sin \mathbf{a}_{pq} & \text{if } p \equiv 1, q \equiv 3 \pmod{4} \\ \sqrt{q} \cdot \sin \mathbf{a}_{pq} & \text{if } p \equiv 3, q \equiv 1 \pmod{4} \\ \sqrt{pq} \cdot \sin \mathbf{a}_{pq} & \text{if } p \equiv q \equiv 3 \pmod{4}, \end{cases} \quad (8)$$

Let $F = \mathbb{Q}(\xi_n)$, where ξ_n is a primitive root of unity. Then the composition field of all double coverings of F/\mathbb{Q} is (see [13, Theorem 1])

$$F(\{\sqrt{u_{pq}}\}_{p < q \in S_n}) \cdot F',$$

where $F' = F(\{\sqrt{-1}\} \bigcup \{\sqrt{l}\}_{l:\text{prime}})$.

3.2. Construction of the Brauer group. Let $K = \mathbb{Q}(\sqrt{d_1 d_2}, \sqrt{d_1 d_3})$ with $d_1, d_2, d_3 \in \mathbb{Z}$ are square-free and relatively prime each other. Without loss generality, we can assume $d_1 d_2 > 0$. In this section, we will explicitly construct the unramified Brauer group of the affine variety X over \mathbb{Q} defined by $N_{K/k}(\Xi) = n$, where $n \in \mathbb{Q}^\times$.

Denote

$$\begin{aligned} S_i &= \{p \text{ rational prime} : p \mid d_i\} \text{ for } 1 \leq i \leq 3, \\ R &= \bigcup_{i < j} S_i \times S_j, \text{ where } 1 \leq i, j \leq 3 \\ N &= \begin{cases} |d_1 d_2 d_3| & \text{if } d_1 d_2 \equiv d_1 d_3 \equiv 1 \pmod{4} \\ 4|d_1 d_2 d_3| & \text{otherwise.} \end{cases} \end{aligned} \tag{9}$$

Let $F = \mathbb{Q}(\xi_N)$. It's clear that K is contained in the cyclotomic field F . For simplicity of the notation, we extend the definition of \mathbf{a}_{pq} and u_{pq} for $p > q$ by

$$\mathbf{a}_{pq} = \mathbf{a}_{qp} \text{ and } u_{pq} = u_{qp}.$$

Let

$$\begin{aligned} \Delta &= \begin{cases} \prod_{(p,q) \in R} \sin \mathbf{a}_{pq} & \text{if } d_1 d_3 > 0 \\ \sqrt{d_1 d_2} \prod_{(p,q) \in R} \sin \mathbf{a}_{pq} & \text{if } d_1 d_3 < 0, \end{cases} \\ L &= F(\sqrt{\Delta}). \end{aligned}$$

Lemma 5. *The field extension L/\mathbb{Q} is Galois and $L \not\subset \mathbb{Q}^{ab}$.*

Proof. If $4 \mid N$, it follows from Theorem 11 and 12 in [6]. So we only need to consider the case $4 \nmid N$, i.e., $d_1 d_2 \equiv d_1 d_3 \equiv 1 \pmod{4}$.

Let $\Delta' = \prod_{(p,q) \in R} u_{pq}$. By an easy computation, we have

$$\Delta' = \prod_{(p,q) \in R} \sin \mathbf{a}_{pq} \prod_{p \mid d_1 d_2 d_3} \sqrt{p^{e_p}},$$

where

$$e_p = \begin{cases} \#\{p \mid d_2 d_3 : p \equiv 3 \pmod{4}\} & \text{if } p \mid d_1 \\ \#\{p \mid d_1 d_3 : p \equiv 3 \pmod{4}\} & \text{if } p \mid d_2 \\ \#\{p \mid d_1 d_2 : p \equiv 3 \pmod{4}\} & \text{if } p \mid d_3. \end{cases}$$

Since $d_1 d_2 > 0$, it implies e_p is even when $p \mid d_2$.

If $d_1 d_3 > 0$, we have $d_2 d_3 > 0$ too. Then e_p is even when $p \mid d_1 d_2$. So $\Delta' = \pm \Delta \cdot u^2$ with $u \in F^\times$. Therefore $L = F(\sqrt{\Delta}) = F(\sqrt{\Delta'})$ or $F(\sqrt{-\Delta'})$. Then L/\mathbb{Q} is Galois and $L \not\subset \mathbb{Q}^{ab}$ by Theorem 1 in [13].

If $d_1 d_3 < 0$, we have $d_2 d_3 < 0$ too. Then e_p is odd when $p \mid d_1 d_2$. So $\Delta' = \pm \Delta \cdot u^2$ with $u \in F^\times$. Therefore $L = F(\sqrt{\Delta}) = F(\sqrt{\Delta'})$ or $F(\sqrt{-\Delta'})$. Then L/\mathbb{Q} is Galois and $L \not\subset \mathbb{Q}^{ab}$ by Theorem 1 in [13]. \square

Theorem 2. *Let X be the affine variety over \mathbb{Q} defined by $N_{K/\mathbb{Q}}(\Xi) = n \in \mathbb{Q}^\times$. Then $\text{Cor}_{K/\mathbb{Q}}(\Xi, \chi)$ generates $\text{Br}(X^c)/\text{Br}_0(X^c)$, where χ is a character of $\text{Gal}(\bar{\mathbb{Q}}/K)$ which factors through $\text{Gal}(L/K)$ and nontrivial on $\text{Gal}(L/F)$.*

Proof. First we show $\text{Cor}_{K/\mathbb{Q}}(\Xi, \chi) \in \text{Br}(X^c)$. Using Theorem 1, we only need to show L/\mathbb{Q} satisfies the condition (*).

Let K' be a subfield of K such that K/K' is cyclic. We want to show $\text{Gal}(L/K')$ is abelian. Since $\text{Gal}(L/K')$ is a quotient of $\text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K')$, we only need to show $\text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K')$ is abelian. The extension $\mathbb{Q}^{ab}(\sqrt{\Delta})/K'$ is Galois by the main theorem in [1]. Let $G^{ab} = \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. There is the central extension

$$\Sigma : 0 \rightarrow \mathbb{Z}/2 \rightarrow \text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/\mathbb{Q}) \rightarrow G^{ab} \rightarrow 0$$

Let

$$\vartheta : G^{ab} \rightarrow \text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/\mathbb{Q})$$

be any set-theoretic splitting of the central extension Σ . Then Σ gives a 2-cocycle

$$a \in Z^2(G^{ab}, \mathbb{Z}/2)$$

defined by the formula

$$a_{\sigma, \tau} = \vartheta(\sigma)\vartheta(\tau)\vartheta(\sigma\tau)^{-1} \quad (\sigma, \tau \in G^{ab}).$$

For each odd prime p , let $G_p \subset G^{ab}$ be the inertia subgroup at p . Let $G_{-1} \subset G^{ab}$ be the subgroup generated by the restriction of complex conjugation to \mathbb{Q}^{ab} . Let $G_2 \subset G^{ab}$ be the subgroup of the inertial subgroup at 2 fixing $\sqrt{-1}$. Let $S = \{-1\} \cup \{p \mid p \text{ is rational prime}\}$. We have

$$G^{ab} = \prod_{p \in S} G_p.$$

For $p \in S$ the profinite group G_p is procyclic. Let $\sigma_p \in G^{ab}$ such that σ_p projects to a topological generator of G_p and projects to 1 in G_q for $q \neq p$. Let $\alpha \in Z^2(G^{ab}, \mathbb{Z}/2)$ defined by $\alpha_{\sigma, \tau} = a_{\sigma, \tau} - a_{\tau, \sigma}$ ($\sigma, \tau \in \text{Gal}(\mathbb{Q}^{ab}/K')$). It's easy to check α is a skew-symmetric (symmetric) bilinear map.

(1) Suppose $d_1 d_3 > 0$. Then $\Delta = \prod_{(p,q) \in T} \text{sin} \mathbf{a}_{pq}$ by our definition. By the Log wedge Formula in §3.4 and §4.3.4 in [1], we have

$$\alpha = \sum_{(p,q) \in R} \delta_{p,q} \in Z^2(G^{ab}, \mathbb{Z}/2),$$

where $\delta_{p,q} = \delta_{q,p} : G^{ab} \times G^{ab} \rightarrow \mathbb{Z}/2$ is defined by

$$((\sigma_l^{i_{l,1}})_{l \in S}, (\sigma_l^{i_{l,2}})_{l \in S}) \mapsto i_{p,1} i_{q,2} + i_{p,2} i_{q,1}$$

and see (9) for the definition of R .

(2) Suppose $d_1 d_3 < 0$. Then $\Delta = \sqrt{d_1 d_2} \prod_{(p,q) \in T} \text{sin} \mathbf{a}_{pq}$. By the Log wedge Formula in §3.4 and §4.3.4 in [1], we have

$$\alpha = \sum_{(p,q) \in R} \delta_{p,q} + \sum_{p \mid d_1 d_2} \delta_{-1,p},$$

where $\delta_{-1,p} : G^{ab} \times G^{ab} \rightarrow \mathbb{Z}/2$ is defined by

$$((\sigma_l^{i_{l,1}})_{l \in S}, (\sigma_l^{i_{l,2}})_{l \in S}) \mapsto i_{-1,1} i_{p,2} + i_{-1,2} i_{p,1}.$$

We have the central extension

$$\Sigma_{K'} : 0 \rightarrow \mathbb{Z}/2 \rightarrow \text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K') \rightarrow \text{Gal}(\mathbb{Q}^{ab}/K') \rightarrow 0$$

Then $\Sigma_{K'}$ gives a 2-cocycle $a' = \text{Res}_{\mathbb{Q}/K'}(a) \in Z^2(\text{Gal}(\mathbb{Q}^{ab}/K'), \mathbb{Z}/2)$. Let

$$\alpha'_{\sigma, \tau} = a'_{\sigma, \tau} - a'_{\tau, \sigma} \quad (\sigma, \tau \in \text{Gal}(\mathbb{Q}^{ab}/K')).$$

We can verify (see [1, Lemma 2.8]) that $\text{Gal}(\mathbb{Q}^{ab}(\sqrt{\Delta})/K')$ is abelian if and only if

$$\alpha'_{\sigma, \tau} = 0 \text{ for any } \sigma, \tau \in \text{Gal}(\mathbb{Q}^{ab}/K')$$

(i) Suppose $d_1 d_3 > 0$. Without loss generality, we can assume $K' = \mathbb{Q}(\sqrt{d_1 d_2})$. Let

$$g_1 = (\sigma_p^{i_{p,1}})_{p \in S}, g_2 = (\sigma_p^{i_{p,2}})_{p \in S} \in \text{Gal}(\mathbb{Q}^{ab}/K') \subset G^{ab}.$$

Since g_1, g_2 fix K' , we have

$$\sum_{p|d_1 d_2} i_{p,j} = \sum_{p|d_1} i_{p,j} + \sum_{p|d_2} i_{p,j} \equiv 0 \pmod{2} \text{ for } j = 1, 2.$$

Then

$$\begin{aligned} \alpha'_{g_1, g_2} &= \alpha_{g_1, g_2} = \sum_{(p,q) \in R} (i_{p,1} i_{q,2} + i_{p,2} i_{q,1}) \\ &= \sum_{(p,q) \in S_1 \times S_2} (i_{p,1} i_{q,2} + i_{p,2} i_{q,1}) + \sum_{(p,q) \in S_1 \times S_3} (i_{p,1} i_{q,2} + i_{p,2} i_{q,1}) \\ &\quad + \sum_{(p,q) \in S_2 \times S_3} (i_{p,1} i_{q,2} + i_{p,2} i_{q,1}) \\ &= \sum_{p|d_1} i_{p,1} \sum_{p|d_2} i_{p,2} + \sum_{p|d_1} i_{p,1} \sum_{p|d_3} i_{p,2} + \sum_{p|d_2} i_{p,1} \sum_{p|d_3} i_{p,2} \\ &\quad + \sum_{p|d_1} i_{p,2} \sum_{p|d_2} i_{p,1} + \sum_{p|d_1} i_{p,2} \sum_{p|d_3} i_{p,1} + \sum_{p|d_2} i_{p,2} \sum_{p|d_3} i_{p,1} \\ &\equiv - \sum_{p|d_2} i_{p,1} \sum_{p|d_2} i_{p,2} - \sum_{p|d_2} i_{p,1} \sum_{p|d_3} i_{p,2} + \sum_{p|d_2} i_{p,1} \sum_{p|d_3} i_{p,2} \\ &\quad - \sum_{p|d_2} i_{p,2} \sum_{p|d_2} i_{p,1} - \sum_{p|d_2} i_{p,2} \sum_{p|d_3} i_{p,1} + \sum_{p|d_2} i_{p,2} \sum_{p|d_3} i_{p,1} \\ &\equiv 0 \pmod{2}. \end{aligned}$$

(ii) Suppose $d_1 d_3 < 0$.

(a) Suppose $K' = \mathbb{Q}(\sqrt{d_1 d_2})$. Let

$$g_1 = (\sigma_p^{i_{p,1}})_{p \in S}, g_2 = (\sigma_p^{i_{p,2}})_{p \in S} \in \text{Gal}(\mathbb{Q}^{ab}/K') \subset G^{ab}.$$

Then we have

$$\sum_{p|d_1 d_2} i_{p,j} \equiv 0 \pmod{2} \text{ for } j = 1, 2.$$

Similar as above one has

$$\begin{aligned} \alpha'_{g_1, g_2} &= \alpha_{g_1, g_2} = \sum_{(p,q) \in R} (i_{p,1} i_{q,2} + i_{p,2} i_{q,1}) + \sum_{p|d_1 d_2} (i_{-1,1} i_{p,2} + i_{-1,2} i_{p,1}) \\ &\equiv 0 + i_{-1,1} \sum_{p|d_1 d_2} i_{p,2} + i_{-1,2} \sum_{p|d_1 d_2} i_{p,1} \\ &\equiv 0 \pmod{2}. \end{aligned}$$

(b) Suppose $K' = \mathbb{Q}(\sqrt{d_1 d_3})$ (similar proof for $K' = \mathbb{Q}(\sqrt{d_2 d_3})$). Let

$$g_1 = (\sigma_p^{i_{p,1}})_{p \in S}, g_2 = (\sigma_p^{i_{p,2}})_{p \in S} \in \text{Gal}(\mathbb{Q}^{ab}/K') \subset G^{ab}.$$

Then we have

$$i_{-1,j} + \sum_{p|d_1 d_3} i_{p,j} \equiv 0 \pmod{2} \text{ for } j = 1, 2.$$

Then

$$\begin{aligned}
\alpha'_{g_1, g_2} &= \alpha_{g_1, g_2} = \sum_{(p, q) \in R} (i_{p,1} i_{q,2} + i_{p,2} i_{q,1}) + \sum_{p|d_1 d_2} (i_{-1,1} i_{p,2} + i_{-1,2} i_{p,1}) \\
&= \sum_{p|d_1} i_{p,1} \sum_{p|d_2} i_{p,2} + \sum_{p|d_1} i_{p,1} \sum_{p|d_3} i_{p,2} + \sum_{p|d_2} i_{p,1} \sum_{p|d_3} i_{p,2} \\
&\quad + \sum_{p|d_1} i_{p,2} \sum_{p|d_2} i_{p,1} + \sum_{p|d_1} i_{p,2} \sum_{p|d_3} i_{p,1} + \sum_{p|d_2} i_{p,2} \sum_{p|d_3} i_{p,1} \\
&\quad + \sum_{p|d_1 d_2} (i_{-1,1} i_{p,2} + i_{-1,2} i_{p,1}) \\
&\equiv - (i_{-1,1} + \sum_{p|d_3} i_{p,1}) \sum_{p|d_2} i_{p,2} - (i_{-1,1} + \sum_{p|d_3} i_{p,1}) \sum_{p|d_3} i_{p,2} + \sum_{p|d_2} i_{p,1} \sum_{p|d_3} i_{p,2} \\
&\quad - (i_{-1,2} + \sum_{p|d_3} i_{p,2}) \sum_{p|d_2} i_{p,1} - (i_{-1,2} + \sum_{p|d_3} i_{p,2}) \sum_{p|d_3} i_{p,1} + \sum_{p|d_2} i_{p,2} \sum_{p|d_3} i_{p,1} \\
&\quad + \sum_{p|d_1 d_2} (i_{-1,1} i_{p,2} + i_{-1,2} i_{p,1}) \\
&\equiv - i_{-1,1} \sum_{p|d_2 d_3} i_{p,2} - i_{-1,2} \sum_{p|d_2 d_3} i_{p,1} + \sum_{p|d_1 d_2} (i_{-1,1} i_{p,2} + i_{-1,2} i_{p,1}) \\
&\equiv i_{-1,1} \sum_{p|d_1 d_3} i_{p,2} + i_{-1,2} \sum_{p|d_1 d_3} i_{p,1} \\
&\equiv - i_{-1,1} i_{-1,2} - i_{-1,2} i_{-1,1} \equiv 0 \pmod{2}.
\end{aligned}$$

Therefore L/K satisfies the condition (*).

Since $Br(X^c)/Br_0(X^c) \cong \mathbb{Z}/2$, we only need to show $Cor_{K/\mathbb{Q}}(\Xi, \chi)$ is nontrivial. Recall

$$F = \mathbb{Q}(\xi_N), K \subset F \subset \mathbb{Q}^{ab} \text{ and } L = F(\sqrt{\Delta}).$$

By Lemma 3, we only need to show χ is not the restriction of a character of $\text{Gal}(\mathbb{Q}/\mathbb{Q})$. Otherwise we assume χ is trivial on $\text{Gal}(\mathbb{Q}/\mathbb{Q}^{ab})$. On the other hand, χ factors through $\text{Gal}(L/K)$. Therefore we have χ is trivial on $\text{Gal}(\bar{\mathbb{Q}}/L \cap \mathbb{Q}^{ab})$. Note that L/\mathbb{Q} is non-abelian (see Lemma 5), hence $F = L \cap \mathbb{Q}^{ab}$. Therefore χ is trivial on $\text{Gal}(\bar{\mathbb{Q}}/F)$, this is a contradiction to that χ factors through $\text{Gal}(L/K)$ and nontrivial on $\text{Gal}(L/F)$. \square

Theorem 3. *Let $m \not\equiv 2 \pmod{4}$. Let $K = \mathbb{Q}(\xi_m)$ be a cyclotomic field and $L = K(\{\sqrt{u_{pq}}\}_{p < q \in S_m})$. Then the 2-torsion subgroup of $\text{Br}(X^c)/\text{Br}_0(X^c)$ is generated by all $Cor_{K/\mathbb{Q}}(\Xi, \chi)$, where χ runs through all characters in the image by the natural map $\text{Hom}(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\text{Gal}(\bar{\mathbb{Q}}/K), \mathbb{Q}/\mathbb{Z})$.*

Proof. Let χ be such a nontrivial character of $\text{Gal}(\bar{\mathbb{Q}}/K)$, then there is a subfield $L' \supset K$ of L with $[L' : K] = 2$ such that χ factors through $\text{Gal}(L'/K)$. And L'/\mathbb{Q} is Galois (non-abelian) by [13, Theorem 1]. That L'/K satisfies the condition (*) follows from the fact:

if the abelian group M is cyclic and G satisfies the central extension

$$0 \rightarrow \mathbb{Z}/2 \rightarrow G \rightarrow M \rightarrow 0,$$

then G is abelian.

Therefore $Cor_{K/\mathbb{Q}}(\Xi, \chi) \in \text{Br}(X^c)$ by Theorem 1.

Let $d = \#S_m$. On the other hand, the Galois group $\text{Gal}(L/K) \cong (\mathbb{Z}/2)^{d(d-1)/2}$ by the linear independent u_{pq} in $K^\times/K^{\times 2}$ (see [13, Lemma 4]). Therefore the subgroup of $\text{Br}(X^c)/\text{Br}_0(X^c)$ generated by all $Cor_{K/\mathbb{Q}}(\Xi, \chi)$ is of 2-rank $d(d-1)/2$. Using the Künneth formula (p. 96 in [8]), we can calculate that the 2-rank of $H^3(\text{Gal}(K/k), \mathbb{Z})$ is also $d(d-1)/2$. Since $\text{Br}(X^c)/\text{Br}_0(X^c) \cong H^3(\text{Gal}(K/k), \mathbb{Z})$, the 2-rank of $\text{Br}(X^c)/\text{Br}_0(X^c)$ is $d(d-1)/2$. Therefore all $Cor_{K/\mathbb{Q}}(\Xi, \chi)$ generate the 2-torsion subgroup of $\text{Br}(X^c)/\text{Br}_0(X^c)$. \square

Finally we will use Theorem 3 to give an explicit example associated to a cyclotomic field. Write $n = 2^{s_1} 7^{s_2} 53^{s_3} p_1^{e_1} \cdots p_g^{e_g} \in \mathbb{Q}^\times$. Let $D(n) = \{p_1, \dots, p_g\}$. Denote

$$D_1 = \{p \in D(n) : \left(\frac{-7}{p}\right) = \left(\frac{53}{p}\right) = -1\}$$

$$D_2 = \{p \in D(n) : \left(\frac{-7}{p}\right) = \left(\frac{53}{p}\right) = 1 \text{ and } \left(\frac{5+2\sqrt{-7}}{p}\right) = -1\}.$$

Example 3. Let $K = \mathbb{Q}(\xi_{7 \cdot 53})$. Then the equation $N_{K/\mathbb{Q}}(\Xi) = n$ is solvable over \mathbb{Q} if and only if the following conditions hold:

- (1) The equation $N_{K/\mathbb{Q}}(\Xi) = n$ is solvable over \mathbb{Q}_p for each p .
- (2) $(-1)^{\sum_{p_i \in D_1} e_i/2 + \sum_{p_i \in D_2} e_i} = (-1)^{s_2} \cdot \left(\frac{n-1}{2}\right)$.

Proof. Let X be the affine variety defined by $N_{K/\mathbb{Q}}(\Xi) = n \in \mathbb{Q}^\times$. We can see

$$\text{Br}(X^c)/\text{Br}_0(X^c) \cong H^3(\text{Gal}(K/k), \mathbb{Z}) \cong \mathbb{Z}/2.$$

Let $L = K(\sqrt{u_{pq}})$. It is easy to verify that $K(\sqrt{5+2\sqrt{-7}})/\mathbb{Q}$ is Galois. Then we have $L = K(\sqrt{5+2\sqrt{-7}})$ by the fact that L is the composition field of all double coverings of K/\mathbb{Q} . Let χ be the unique nontrivial character of $\text{Gal}(\bar{\mathbb{Q}}/K)$ which factors through $\text{Gal}(L/K)$. Then $Cor_{K/\mathbb{Q}}(\Xi, \chi)$ is the unique generator of $\text{Br}(X^c)/\text{Br}_0(X^c)$ by Theorem 3. \square

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